LIBERTY PAPER SET

STD. 12 : Mathematics

Full Solution

Time : 3 Hours **ASSIGNMENT PAPER 5**

PART A

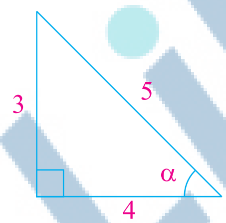
1. (A) **2.** (C) **3.** (A) **4.** (C) **5.** (B) **6.** (A) **7.** (D) **8.** (C) **9.** (B) **10.** (D) **11.** (D) **12.** (A) **13.** (C)
14. (C) **15.** (A) **16.** (A) **17.** (B) **18.** (D) **19.** (A) **20.** (B) **21.** (B) **22.** (C) **23.** (C) **24.** (A) **25.** (B)
26. (A) **27.** (B) **28.** (C) **29.** (C) **30.** (B) **31.** (C) **32.** (B) **33.** (D) **34.** (B) **35.** (D) **36.** (A) **37.** (C)
38. (C) **39.** (A) **40.** (C) **41.** (D) **42.** (B) **43.** (A) **44.** (D) **45.** (C) **46.** (C) **47.** (A) **48.** (B) **49.** (C)
50. (D)

PART B

SECTION A

1.

Method 1 :



Let, $\sin^{-1}\frac{3}{5} = \alpha$

$\therefore \quad \frac{3}{5} = \sin\alpha$

$\therefore \quad \tan\alpha = \frac{3}{4}$

Similarly,

Let, $\cos^{-1}\frac{3}{2} = \beta$

$\therefore \quad \frac{3}{2} = \cot\beta$

$\therefore \quad \tan\beta = \frac{2}{3}$

Now, $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

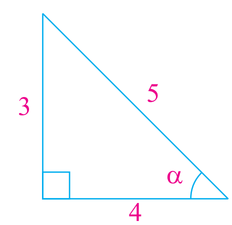
$$= \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}$$

$$= \frac{9+8}{12-6}$$

$$= \frac{17}{6}$$

Method 2 :



$\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

Here, $\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$

$$= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right] \quad \left(\frac{3}{4}\cdot\frac{2}{3} < 1\right)$$

$$= \tan\left[\tan^{-1}\left(\frac{9+8}{12-6}\right)\right]$$

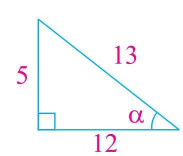
$$= \tan\left[\tan^{-1}\left(\frac{17}{6}\right)\right]$$

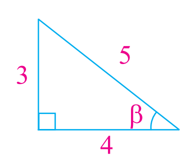
$$= \frac{17}{6}$$

2.

L.H.S. $= \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$

$\cos^{-1}\frac{12}{13} = \alpha \quad , \quad \sin^{-1}\frac{3}{5} = \beta$





$\therefore\ \cos\alpha = \frac{12}{13}, \sin\alpha = \frac{5}{12}$ | $\sin\beta = \frac{3}{5}, \cos\beta = \frac{4}{5}$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$$

$$= \frac{20}{65} + \frac{36}{65}$$

$$= \frac{56}{65}$$

$$\therefore\quad \alpha + \beta = \sin^{-1}\frac{56}{65}$$

$$\therefore\quad \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$$

3.

Differentiate w.r.t. x,

$$\therefore\ 2x + x \cdot \frac{dy}{dx} + y + 2y \cdot \frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx}(x + 2y) = -2x - y$$

$$\therefore\ \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

4.

$$I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\, dx$$

Take, $\tan x = t$,

$\therefore\ \sec^2 x\, dx = dt$

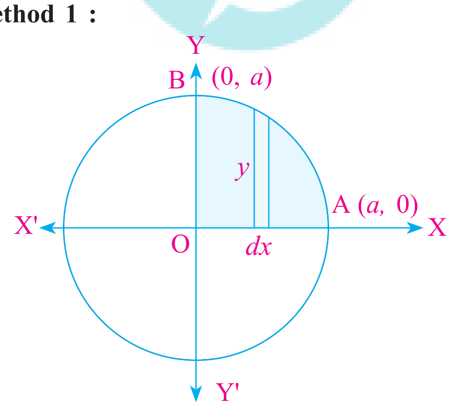
$$\therefore\ I = \int \frac{dt}{\sqrt{t^2 + (2)^2}}$$

$$= \log\left|t + \sqrt{t^2 + 4}\right| + c$$

$$\therefore\ I = \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + c$$

5.

Method 1 :



From Fig, the whole area enclosed by the given circle = 4 (area of the region AOBA bounded by the curve, x-axis and the ordinates $x = 0$ and $x = a$) [as the circle is symmetrical about both x-axis and y-axis]

$$= 4\int_0^a y\, dx \quad \text{(taking vertical strips)}$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

Since, $x^2 + y^2 = a^2$ gives $y = \pm\sqrt{a^2 - x^2}$

As the region AOBA lies in the first quadrant, y is taken as positive. Integrating, we get the whole area enclosed by the given circle

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

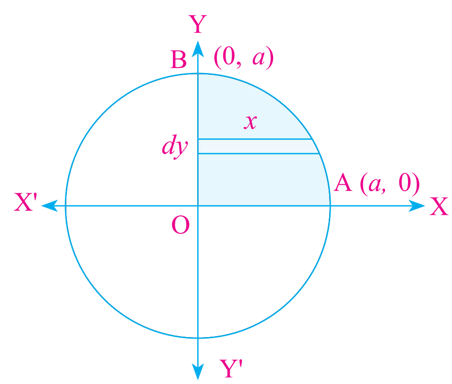
$$= 4\left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= 4\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}\right)$$

$= \pi a^2$ sq. unit

Method 2 :

considering horizontal strips as shown in Fig, the whole area of the region enclosed by circle



$$= 4\int_0^a x\, dy$$

$$= 4\int_0^a \sqrt{a^2 - y^2}\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

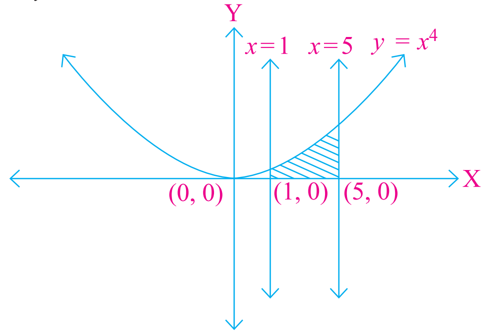
$$= 4\left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= 4\,\frac{a^2}{2}\,\frac{\pi}{2}$$

$= \pi a^2$ sq. unit

6.

$x^4 = y$



Required area,

$A = |I|$

$\therefore \; I = \int_1^5 y \, dx$

$\therefore \; I = \int_1^5 x^4 \, dx$

$\therefore \; I = \left[\frac{x^5}{5}\right]_1^5$

$\therefore \; I = \frac{1}{5}\left((5)^5 - 1\right)$

$\therefore \; I = \frac{1}{5}(3125 - 1)$

$\therefore \; I = 624.8$

Now, $A = |I|$

$= |624.8|$

$\therefore \quad A = 624.8$ sq. units

7.

$$\frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\therefore \; \frac{dy}{1+y^2} = (1 + x^2)\, dx$$

→ Integrate both the sides,

$$\therefore \; \int \frac{dy}{y^2+1} = \int (x^2 + 1)\, dx$$

$$\therefore \; tan^{-1}y = \left(\frac{x^3}{3} + x\right) + c$$

$$\therefore \; tan^{-1}y = \frac{x^3}{3} + x + c;$$

Which is required general solution of given differential equation.

8.

The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

$$= \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$$

$$= \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\vec{a}$ is

$$7\hat{a} = 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right)$$

$$= \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

9.

$$L : \frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$$

$$\therefore \; \frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2}$$

$$L : \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda\left(-3\hat{i} + \frac{2p}{7}\hat{j} + 2\hat{k}\right)$$

$$\therefore \; \vec{b_1} = -3\hat{i} + \frac{2p}{7}\hat{j} + 2\hat{k}$$

$$\text{Now, } \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

$$\therefore \; \frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

$$M : \vec{r} = (\hat{i} + 5\hat{j} + 6\hat{k}) + \mu\left(\frac{-3p}{7}\hat{i} + \hat{j} - 5\hat{k}\right)$$

$$\therefore \; \vec{b_2} = \frac{-3p}{7} + \hat{j} - 5\hat{k}$$

→ L and M are perpendicular to each other,

$$\vec{b_1} \cdot \vec{b_2} = 0$$

$$\therefore \; \left(-3\hat{i} + \frac{2p}{7}\hat{j} + 2\hat{k}\right)\cdot\left(-\frac{3p}{7}\hat{i} + \hat{j} - 5\hat{k}\right) = 0$$

$$\therefore \; \frac{9p}{7} + \frac{2p}{7} - 10 = 0$$

$$\therefore \; \frac{11p}{7} = 10$$

$$\therefore \; p = \frac{70}{11}$$

10.

$A(\vec{a}) = -2\hat{i} + 4\hat{j} - 5\hat{k}$ line on the line

line is parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Direction of line $\vec{b} = 3\hat{i} + 5\hat{j} + 6\hat{k}$

∴ Equation of parallel line is

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{l_2} = \frac{z-z_1}{l_3}$$

$$\frac{x-(-2)}{3} = \frac{y-4}{5} = \frac{z-(-5)}{6}$$

$$\therefore \; \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$$

Which is required cartesian equation of line.

11.

From a box containing 10 black and 8 red balls with replacement,

$$= 2 \times \frac{^{10}C_1}{^{18}C_1} \times \frac{^{8}C_1}{^{18}C_1} = \frac{40}{81}$$

12.

$P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$, $P(E \cap F) = \frac{1}{5}$

$$\therefore\ P(E) \cdot P(F) = \frac{3}{5} \times \frac{3}{10}$$
$$= \frac{9}{50}$$
$$\neq P(E \cap F)$$

$\therefore$ E and F are not independent events.

SECTION B

13.

Here $f : N \to N$, $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

Take, $n_1 = 3$, $n_2 = 4$,

$f(n_1) = \frac{3+1}{2} = 2$ and $f(n_2) = f(4) = \frac{4}{2} = 2$

Here, $n_1 \neq n_2$ but $f(n_1) = f(n_2)$

$\therefore$ f is not one-one function.

Domain N = {1, 2, 3, 4, 5, 6,......}

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

$f(1) = \frac{1+1}{2} = 1$

$f(2) = \frac{2}{2} = 1$

$f(3) = \frac{3+1}{2} = 2$

$f(4) = \frac{4}{2} = 2$

$f(5) = \frac{5+1}{2} = 3$

$f(6) = \frac{6}{2} = 3$ .....

$\therefore$ R_f = {1, 2, 3, 4 ....} = N (co-domain)

$\therefore$ f is onto function.

14.

$$A^2 = A \cdot A$$
$$= \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

Now, $A^2 = KA - 2I$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = K \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3K & -2K \\ 4K & -2K \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3K-2 & -2K \\ 4K & -2K-2 \end{bmatrix}$$

$\therefore\ 1 = 3K - 2$	$-2 = -2K$	$4 = 4K$	$-4 = -2K + 2$
$\therefore\ 3K = 3$	$\therefore\ K = 1$	$\therefore\ K = 1$	$-2 = -2K$
$K = 1$			$\therefore\ K = 1$

$\therefore$ K = 1

15.

We have $A^2 = A \cdot A \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence, $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= O$$

Now, $A^2 - 4A + I = O$

Therefore, $AA - 4A = -I$

OR $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$

(Post multiplying by A^{-1} because $|A| \neq 0$)

OR $A(AA^{-1}) - 4I = -A^{-1}$

OR $AI - 4I = -A^{-1}$

OR $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

16.

Suppose, $u = (sinx)^x$ and $v = sin^{-1}\sqrt{x}$

$\therefore\ y = u + v$

Now, differentiate w.r.t. x,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{... (1)}$$

Now, $u = (sinx)^x$

Take both the sides *log*,

$log\ u = x log\ sinx$

Now, differentiate w.r.t. x,

$$\frac{1}{u}\frac{du}{dx} = x \frac{d}{dx} log\ sinx + log\ sinx \frac{d}{dx} x$$
$$= x \times \frac{1}{sinx} cosx + log\,sinx$$
$$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot cotx + logsinx$$

$$\frac{du}{dx} = u[x \cdot cotx + logsinx]$$

$$\frac{du}{dx} = (sinx)^x \; [x \cdot cotx + logsinx] \quad \text{... (2)}$$

Now, $v = sin^{-1}\sqrt{x}$

Differentiate w.r.t. x,

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \quad \text{... (3)}$$

Put the value of equation (2) and (3) in equation (1),

$$\frac{dy}{dx} = (sinx)^x\left[xcotx + logsinx\right] + \frac{1}{2\sqrt{x-x^2}}$$

17.

$$y = log(1+x) - \frac{2x}{2+x}, \; x > -1$$

$$\frac{dy}{dx} = \frac{1}{1+x} - \left[\frac{(2+x)(2)-(2x)}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \left[\frac{4+2x-2x}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+4x+x^2-4-4x}{(1+x)(2+x)^2}$$

$$\frac{dy}{dx} = \frac{x^2}{(1+x)(2+x)^2}$$

Now, $x > -1 \Rightarrow \; x^2 \geq 0$

$\Rightarrow \; (1+x) > 0$

$\Rightarrow \; (2+x)^2 > 0$

$\Rightarrow \; \frac{dy}{dx} \geq 0$

Therefore, $x > -1$ is an increasing function throughout its domain.

18.

Method 1 :

Suppose, $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

Now, $\vec{d} \perp \vec{a}$

$\therefore \; \vec{d} \cdot \vec{a} = 0$

$\therefore \; (d_1\hat{i} + d_2\hat{j} + d_3\hat{k}) \cdot (\hat{i} + 4\hat{j} + 2\hat{k}) = 0$

$\therefore \; d_1 + 4d_2 + 2d_3 = 0$ .... (1)

Now, $\vec{d} \perp \vec{b}$

$\therefore \; \vec{d} \cdot \vec{b} = 0$

$\therefore \; (d_1\hat{i} + d_2\hat{j} + d_3\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 7\hat{k}) = 0$

$\therefore \; 3d_1 - 2d_2 + 7d_3 = 0$ .... (2)

Now, $\vec{c} \cdot \vec{d} = 15$

$\therefore \; (2\hat{i} - \hat{j} + 4\hat{k})(d_1\hat{i} + d_2\hat{j} + d_3\hat{k}) = 15$

$\therefore \; 2d_1 - d_2 + 4d_3 = 15$ .... (3)

Solving equation (1) and (2),

$$d_1 + 4d_2 + 2d_3 = 0$$
$$6d_1 - 4d_2 + 14d_3 = 0$$
$$7d_1 + 16d_3 = 0 \quad \text{......... (4)}$$

Solving equation (2) and (3),

$$3d_1 - 2d_2 + 7d_3 = 0$$
$$\underset{-}{4d_1} \underset{+}{-\,2d_2} + \underset{-}{8d_3} = \underset{-}{30}$$
$$-d_1 - d_3 = -30$$
$$\therefore d_1 + d_3 = 30 \quad \text{......... (5)}$$

Solving equation (4) and (5),

$$7d_1 + 16d_3 = 0$$
$$\underset{-}{7d_1} + \underset{-}{7d_3} = \underset{-}{210}$$
$$9d_3 = -210$$
$$\therefore d_3 = -\frac{70}{3}$$

Put the value of d_3 in equation (5),

$$d_1 - \frac{70}{3} = 30$$

$$\therefore \quad d_1 = \frac{90+70}{3}$$

$$\therefore \quad d_1 = \frac{160}{3}$$

Put the value of d_1 and d_3 in equation (1),

$$\frac{160}{3} + 4d_2 - \frac{140}{3} = 0$$

$$\therefore \; 4d_2 = -\frac{20}{3}$$

$$\therefore \; d_2 = \frac{-5}{3}$$

$$\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$$

$$\therefore \; \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k}$$

Method 2 :

$\vec{d}$ is perpendicular to both vector $\vec{a}$ and $\vec{b}$

$\therefore \; \vec{d}$ is parallel to $\vec{a} \times \vec{b}$

$\therefore \; \vec{d} = p \cdot (\vec{a} \times \vec{b})$

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \hat{i} \cdot (28+4) - \hat{j}(7-6) + \hat{k}(-2-12)$$

$$= 32\hat{i} - \hat{j} - 14\hat{k}$$

Now, $\vec{d} = p(\vec{a} \times \vec{b})$

$= p(32\hat{i} - \hat{j} - 14\hat{k})$ .... (1)

and $\vec{c} \cdot \vec{d} = 15$

$\therefore (2\hat{i} - \hat{j} + 4\hat{k}) \cdot [p(32\hat{i} - \hat{j} - 14\hat{k})] = 15$

$\therefore p[(2)(32) + (-1)(-1) + 4(-14)] = 15$

$\therefore 9p = 15$

$\therefore p = \frac{15}{9} = \frac{5}{3}$

Put the value of $p = \frac{5}{3}$ in equation (1),

$\vec{d} = \frac{5}{3}(32\hat{i} - \hat{j} - 14\hat{k})$

$= \frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k})$

19.

Comparing (1) and (2) with $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ respectively,

$\vec{a_1} = \hat{i} + \hat{j}$, $\vec{b_1} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{a_2} = 2\hat{i} + \hat{j} - \hat{k}$

and $\vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k}$

Therefore, $\vec{a_2} - \vec{a_1} = \hat{i} - \hat{k}$

and $\vec{b_1} \times \vec{b_2} = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$= 3\hat{i} - \hat{j} + 7\hat{k}$

So, $|\vec{b_1} \times \vec{b_2}| = \sqrt{9+1+49}$

$= \sqrt{59}$

Hence, the shortest distance between the given lines is given by

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{3 - 0 + 7}{\sqrt{59}} \right| = \frac{10}{\sqrt{59}} \text{ unit}$$

20.

$3x + 5y \leq 15$

$5x + 2y \leq 10$

$x \geq 0$

$y \geq 0$

Objective function $Z = 5x + 3y$

$3x + 5y = 15$ ... (i)

x	0	5
y	3	0

$5x + 2y = 10$ ... (ii)

x	0	2
y	5	0

Solving equation (i) and (ii),

$6x + 10y = 30$

$25x + 10y = 50$

$-\quad - \quad -$

$19x = 20$

$\therefore x = \frac{20}{19}$

$5y = -\frac{60}{19} + 15$

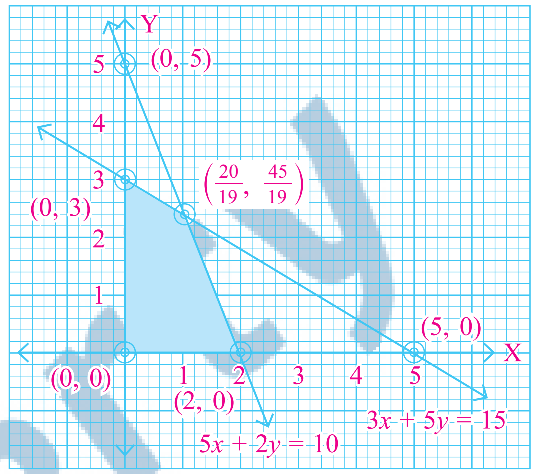
$\therefore y = -\frac{12}{19} + 3$

$= \frac{-12+57}{19}$

$\therefore y = \frac{45}{19}$

$\left(\frac{20}{19}, \frac{45}{19}\right)$

(0, 0)



The shaded region in fig. is feasible region determined by the system if constraints which is bounded. The co-ordinates of corner point (0, 0), (2, 0), $\left(\frac{20}{19}, \frac{45}{19}\right)$ and (0, 3).

Corner Point	Corresponding value of $Z = 5x + 3y$
(0, 3)	9
(2, 0)	10
(0, 0)	0
$\left(\frac{20}{19}, \frac{45}{19}\right)$	$\frac{100+135}{19} = \frac{235}{19}$ ← Maximum

Thus, the max. value of Z is $\frac{235}{19}$ at point $\left(\frac{20}{19}, \frac{45}{19}\right)$.

21.

Event E_1 : Machine A produced items

Event E_2 : Machine B produced items

$P(E_1) = \frac{60}{100}$

$P(E_2) = \frac{40}{100}$

Event A : Item is found to be defective

$P(A \mid E_1) = 0.02 = \frac{2}{100}$,

$P(A \mid E_2) = 0.01 = \frac{1}{100}$

Item is found to be defective and produced by machine B, probability,

$$\therefore\ P(E_2 \mid A) = \frac{P(E_2)\cdot P(A|E_2)}{P(A)}$$

$$\therefore\ P(A) = P(E_1)\cdot P(A \mid E_1) + P(E_2)\cdot P(A \mid E_2)$$

$$= \frac{60}{100}\times\frac{2}{100}+\frac{40}{100}\times\frac{1}{100}$$

$$= \frac{120}{10000}+\frac{40}{10000}$$

$$= \frac{160}{10000}$$

$$\therefore\ P(E_2 \mid A) = \frac{P(A|E_2)\cdot P(E_2)}{P(A)}$$

$$= \frac{\frac{40}{100}\times\frac{1}{100}}{\frac{160}{1000}}$$

$$= \frac{1}{4}$$

SECTION C

22.

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$$

Now, A'A = I

$$\therefore \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0+4y^2+z^2 & 0+2y^2-z^2 & 0-2y^2+z^2 \\ 0+2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ 0-2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore\ 4y^2 + z^2 = 1$ ... (1)

$2y^2 - z^2 = 0$ ... (2)

$x^2 + y^2 + z^2 = 1$ ... (3)

$x^2 - y^2 - z^2 = 0$ ... (4)

Solving equation (1) and (2),

$4y^2 + z^2 = 1$

$2y^2 - z^2 = 0$

$6y^2 = 1$

$$\therefore\ y^2 = \frac{1}{6}$$

$$\therefore\ y = \pm\frac{1}{\sqrt{6}}$$

From equation (2), $2 \times \frac{1}{6} - z^2 = 0$ we get,

$$\therefore\ z^2 = \frac{1}{3}$$

$$\therefore\ z = \pm\frac{1}{\sqrt{3}}$$

Put $z^2 = \frac{1}{3}$, $y^2 = \frac{1}{6}$ in equation (3),

$$x^2 + \frac{1}{6} + \frac{1}{3} = 1$$

$$\therefore\ x^2 = 1 - \frac{1}{6} - \frac{1}{3}$$

$$= \frac{6-1-2}{6}$$

$$= \frac{1}{2}$$

$$\therefore\ x = \pm\frac{1}{\sqrt{2}}$$

Therefore, $x = \pm\frac{1}{\sqrt{2}}$,

$y = \pm\frac{1}{\sqrt{6}}$,

$z = \pm\frac{1}{\sqrt{3}}$

23.

Suppose, $\frac{1}{x} = a$, $\frac{1}{y} = b$, $\frac{1}{z} = c$

$2a + 3b + 10c = 4$

$4a - 6b + 5c = 1$

$6a + 9b - 20c = 2$

The equation can be represented as matrix form,

$$\therefore \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ AX = B

Where, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

AX = B

$\therefore\ X = A^{-1}B$

For finding A^{-1},

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 2(75) - 3(-110) + 10(72)$

$= 150 + 330 + 720$

$= 1200 \neq 0$

$\therefore\ A^{-1}$ exists.

For finding *adj* A,

Co-factor of element 2 $A_{11} = (-1)^2 \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix}$

$= 1(120 - 45)$

$= 75$

Co-factor of element 3 $A_{12} = (-1)^3 \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix}$

$= (-1)(-80 - 30)$

$= 110$

Co-factor of element 10 $A_{13} = (-1)^4 \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$

$= 1(36 + 36)$

$= 72$

Co-factor of element 4 $A_{21} = (-1)^3 \begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix}$

$= (-1)(-60 - 90)$

$= 150$

Co-factor of element –6 $A_{22} = (-1)^4 \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix}$

$= 1(-40 - 60)$

$= -100$

Co-factor of element 5 $A_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix}$

$= (-1)(18 - 18)$

$= 0$

Co-factor of element 6 $A_{31} = (-1)^4 \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix}$

$= 1(15 + 60)$

$= 75$

Co-factor of element 9 $A_{32} = (-1)^5 \begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix}$

$= (-1)(10 - 40)$

$= 30$

Co-factor of element –20 $A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix}$

$= 1(-12 - 12)$

$= -24$

$$adj\ A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$X = A^{-1}B$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\therefore \frac{1}{x} = \frac{1}{2},\ \frac{1}{y} = \frac{1}{3},\ \frac{1}{z} = \frac{1}{5}$$

Solution : $x = 2,\ y = 3,\ z = 5$

24.

$(x - a)^2 + (y - b)^2 = c^2$ .... (1)

Now, differentiate w.r.t. x,

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0$$

$$\therefore (x - a) + (y - b)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-(x-a)}{y-b} \quad \text{.......... (2)}$$

Now, differentiate w.r.t. x,

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)(1)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)+\frac{(x-a)(x-a)}{(y-b)}}{(y-b)^2}\right] \quad (\because \text{ From equation (2)})$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-c^2}{(y-b)^3} \quad (\because \text{ From equation (1)})$$

$$\text{Now, } \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{-[c^2]^{\frac{3}{2}}}{[(y-b)^2]^{\frac{3}{2}}} \times \frac{(y-b)^3}{[c^2]}$$

$$= \frac{-c^3}{(y-b)^3} \times \frac{(y-b)^3}{c^2}$$

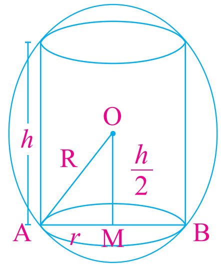
$$= -c, \quad c > 0$$

Which is constant independent of a and b.

25.

Here, radius of sphere is R,

Suppose, radius and height of cylinder are r and h.



From right angle triangle ΔOMA,

$$R^2 = r^2 + \frac{h^2}{4} \quad \text{........... (1)}$$

→ Volume of cylinder (V)= $\pi r^2 h$

$$\therefore \quad V = \pi\left(R^2 - \frac{h^2}{4}\right)(h)$$

$$\therefore \quad f(h) = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

$$f'(h) = \pi\left(R^2 - \frac{3h^2}{4}\right)$$

$$f''(h) = \pi\left(0 - \frac{6h}{4}\right)$$

$$f''(h) = \frac{-3\pi h}{2} < 0$$

→ For finding maximum volume of cylinder

$$f'(h) = 0$$

$$\therefore \quad \pi\left(R^2 - \frac{3h^2}{4}\right) = 0$$

$$\therefore \quad R^2 = \frac{3h^2}{4}$$

$$\therefore \quad R = \frac{\sqrt{3}\,h}{2}$$

$$\therefore \quad h = \frac{2R}{\sqrt{3}}$$

→ Volume of cylinder $V = \pi\left(R^2 - \frac{h^2}{4}\right)(h)$

$$= \pi\left(R^2 - \frac{4R^2}{4(3)}\right)\left(\frac{2R}{\sqrt{3}}\right)$$

$$= \pi\left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right)$$

$$= \frac{4\pi R^3}{3\sqrt{3}}$$

26.

$$I = \int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$$

$$5x + 3 = A\frac{d}{dx}(x^2 + 4x + 10) + B$$

$$5x + 3 = A(2x + 4) + B$$

$$5x + 3 = 2Ax + 4A + B$$

→ Comparing co-efficient of x and constant term,

$\therefore$ 2A = 5	$\therefore$ 4A + B = 3
$\therefore$ A = $\frac{5}{2}$	$\therefore$ $4\left(\frac{5}{2}\right)$ + B = 3
	$\therefore$ 10 + B = 3
	$\therefore$ B = –7

$$= \int \frac{\frac{5}{2}\cdot\frac{d}{dx}(x^2+4x+10) - 7}{\sqrt{x^2+4x+10}}\,dx$$

$$= \frac{5}{2}\int \frac{\frac{d}{dx}(x^2+4x+10)}{\sqrt{x^2+4x+10}}\,dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}\,dx$$

$$= \frac{5}{2}\int (x^2+4x+10)^{-\frac{1}{2}}\,\frac{d}{dx}(x^2+4x+10)\,dx - 7\int \frac{dx}{\sqrt{x^2+2(2x)+4-4+10}}$$

$$= \frac{5}{2}\int (x^2+4x+10)^{-\frac{1}{2}}\,\frac{d}{dx}(x^2+4x+10)\,dx - 7\int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$$

$$= \frac{5}{2}\int \frac{(x^2+4x+10)^{\frac{1}{2}}}{\frac{1}{2}} - 7\log\left|x + 2 + \sqrt{x^2+4x+10}\right| + c$$

$$\therefore \quad I = 5\sqrt{x^2+4x+10} - 7\log\left|x + 2 + \sqrt{x^2+4x+10}\right| + c$$

27.

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$$

$$\therefore \quad dy = \frac{(2x^2+x)}{(x^3+x^2+x+1)}\,dx$$

$$\therefore \quad dy = \frac{(2x^2+x)\,dx}{x^2(x+1)+1(x+1)}$$

$$\therefore \quad dy = \frac{(2x^2+x)\,dx}{(x^2+1)(x+1)}$$

→ Integrate both the sides,

$$\therefore \quad \int 1\,dy = \int \frac{(2x^2+x)\,dx}{(x^2+1)(x+1)}$$

Now, $\dfrac{2x^2 + x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+1}$

$\therefore\ 2x^2 + x = A(x^2 + 1) + (Bx + C)(x + 1)$

Take, $x = -1$

$\therefore\ 2(-1)^2 - 1 = A(2) + (Bx + C)(0)$

$\therefore\ 2 - 1 = 2A$

$\therefore\ A = \dfrac{1}{2}$

Take, $x = 0$

$\therefore\ 0 = A(1) + B(0) + C(1)$

$\therefore\ 0 = \dfrac{1}{2} + C$

$\therefore\ C = -\dfrac{1}{2}$

Take, $x = 1$

$\therefore\ 2(1) + 1 = A(2) + (B + C)(2)$

$\therefore\ 3 = 2A + 2B + 2C$

$\therefore\ 3 = 2\left(\dfrac{1}{2}\right) + 2B + 2\left(-\dfrac{1}{2}\right)$

$\therefore\ 3 = 1 + 2B - 1$

$\therefore\ B = \dfrac{3}{2}$

$$\therefore\ \int 1\, dy = \frac{1}{2}\int \frac{dx}{x+1} + \int \frac{\left(\frac{3}{2}x - \frac{1}{2}\right)}{x^2+1}\, dx$$

$$\therefore\ \int 1\, dy = \frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{x\, dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\therefore\ \int 1\, dy = \frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{4}\int \frac{2x\, dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\therefore\ \int 1\, dy = \frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{4}\int \frac{\frac{d}{dx}(x^2+1)}{x^2+1}\, dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\therefore\ y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + k \quad \ldots (1)$$

For particular solution,

Now, $y = 1$ when $x = 0$,

$\therefore\ 1 = \dfrac{1}{2}\log|1| + \dfrac{3}{4}\log|1| - \dfrac{1}{2}\tan^{-1}(0) + c$

$\therefore\ 1 = 0 + 0 + 0 + k$

$\therefore\ k = 1$

Putting the value of k in equation (1),

$$\therefore\ y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + 1$$

$$\therefore\ y = \frac{2}{4}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + 1$$

$$\therefore\ y = \frac{1}{4}[\log(x+1)^2 + \log(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1$$

$$\therefore\ y = \frac{1}{4}[\log[(x+1)^2(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1;$$

Which is required particular solution of given differential equation.